# ISOPERIMETRIC CONSTANTS FOR PRODUCT PROBABILITY MEASURES *

S. G. Bobkov† and C. Houdré ‡

July 24, 1995

**Abstract**

A dimension free lower bound is found for isoperimetric constants of product probability measures. From this, some analytic inequalities are derived.

## 1 Introduction

Let $(X, d)$ be a metric space equipped with a non–atomic, separable Borel probability measure $\mu$. In the present paper we study the quantity

$$Is(\mu) = \inf \frac{\mu^+(A)}{\min(\mu(A), 1 - \mu(A))} \tag{1.1}$$

---

*Key words: Isoperimetry, Poincaré Inequalities, Cheeger's inequality, Khinchine–Kahane inequality, Hölder's Inequality

†Research supported in part by the ISF grant NZX000 and NZX300. This author enjoyed the hospitality of the Faculty of Wiskunde and Informatica, Free University of Amsterdam, while part of this research was carried out.

‡Research supported in part by an NSF Postdoctoral Fellowship. This author enjoyed the hospitality of Le Cermics, ENPC, France, of the Steklov Mathematical Institute (Sankt Petersburg branch) and of the Department of Mathematics, University of Syktyvkar, Russia, while part of this research was carried out.

which was introduced by Cheeger [C] in a Riemannian geometry context. The infimum in (1.1) is taken over all Borel sets $A \subset X$ of measure $0 < \mu(A) < 1$, and $\mu^+$ denotes the surface measure of $A$, i.e.,

$$\mu^+(A) = \liminf_{h \to 0^+} \frac{\mu(A^h) - \mu(A)}{h}$$

where $A^h = \{x \in X : d(x,a) < h \text{ for some } a \in A\}$ is the open $h$–neighbourhood of $A$ (for the metric $d$). For any function $f : X \to \mathbf{R}$, we also define the modulus of its gradient

$$|\nabla f(x)| = \limsup_{d(x,y) \to 0^+} \frac{|f(x) - f(y)|}{d(x,y)}.$$

The space $X^n = X \times \cdots \times X$ is endowed with the metric $d_n$ given by $d_n(x,y) = \left(\sum_{k=1}^n d^2(x_k, y_k)\right)^{1/2}$ and with the probability measure $\mu^n$ which is the $n$–fold tensor product of $\mu$ with itself. Also, to avoid pathologies, we assume throughout that for any Lipschitz function $f$ on $(X^n, d_n)$, $|\nabla f|^2 = \sum_{k=1}^n |\nabla_{x_k} f|^2$ almost everywhere (with respect to $\mu^n$). On the Euclidean space $\mathbf{R}^n$, and via Rademacher Theorem, this standing assumption holds for any absolutely continuous probability measure.

With these notations, our main result can be stated as follows:

**Theorem 1.1** *For any triple $(X, d, \mu)$ as above,*

$$Is(\mu^n) \geq \frac{1}{4\sqrt{3}} Is(\mu), \tag{1.2}$$

*for all $n = 1, 2, \cdots$. Equivalently, and up to a universal constant, for any function $f : X^n \to [0,1]$ which has finite Lipschitz constant on every ball in $(X^n, d_n)$*

$$K\mathbf{Var}(f) \leq \mathbf{E} \min\left(\frac{1}{Is(\mu)}|\nabla f|, \frac{1}{Is^2(\mu)}|\nabla f|^2\right). \tag{1.3}$$

*Above, the expectation and the variance are taken with respect to $\mu^n$, and one can take $K = 1/288$.*

From (1.3), $KIs(\mu)\mathbf{Var}(f) \leq \mathbf{E}|\nabla f|$, and approximating the indicator function $\mathbf{1}_A$ by Lipschitz functions $f_k$ so that to have $\liminf_k \mathbf{E}|\nabla f_k| \leq (\mu^n)^+(A)$, this gives

$$(\mu^n)^+(A) \geq KIs(\mu)\mathbf{Var}(\mathbf{1}_A) \geq \frac{1}{2}KIs(\mu)\min(\mu(A), 1-\mu(A)).$$

Therefore, (1.3) implies (1.2) with a worse but still universal constant.

One of the most interesting partial cases of Theorem 1.1, is when the measure $\mu$ is the double exponential distribution on the real line $\nu(dx) = 1/2\exp(-|x|)dx$. In this case, it is known (Talagrand [T]) that $\nu$ satisfies the isoperimetric inequality

$$\nu^+(A) \geq \min(\nu(A), 1-\nu(A)), \tag{1.4}$$

with equality for the intervals $A = (-\infty, x]$, and thus, $Is(\nu) = 1$. It is then natural to ask whether or not, (1.4) continues to hold for the product measure $\nu^n$ with a (multiplicative) constant independent of the dimension, i.e., whether or not $\inf_n Is(\nu^n) > 0$. In other words, one can ask whether or not $\nu^n$ satisfies an $L^1$–Poincaré type inequality with a dimension free constant, i.e., whether or not for all smooth functions $f$ on $X^n$ with $\mathbf{E}f = 0$

$$K\mathbf{E}|f| \leq \mathbf{E}|\nabla f|. \tag{1.5}$$

Theorem 1.1 gives a positive answer to this question and in fact:

**Theorem 1.2** *Let $\mu$ be a probability measure on the real line $\mathbf{R}$ with a positive continuous density concentrated on some interval (finite or not). The measure $\mu^n$ satisfies (1.5) with some constant independent of the dimension if and only if the increasing map which transforms the double exponential measure $\nu$ into $\mu$ has finite Lipschitz constant.*

As easily seen in terms of the distribution function $F$ and of the density $p$ of $\mu$, this last property can be expressed as

$$\sup_{a<x<b} \frac{\min(F(x), 1-F(x))}{p(x)} < +\infty, \tag{1.6}$$

where $a = \inf\{x : F(x) > 0\}, b = \sup\{x : F(x) < 1\}$. It is known (see Borovkov and Utev [BU]), that this last condition is sufficient for $\mu$ to satisfy the $L^2$–Poincaré type inequality

$$K\mathbf{E}|f|^2 \leq \mathbf{E}|\nabla f|^2, \tag{1.7}$$

where $\mathbf{E}f = 0$ (and where $K$ is independent of the dimension by the additivity property of (1.7) (see Gross [G] for the log–Sobolev version of this property)). On the other hand, as can be seen from a recent characterization due to Chen and Lou [CL], (1.7) does not imply (1.6). Therefore, the family of probability measures which satisfies an $L^2$–Poincaré type inequality is larger than the family of probability measures satisfying (1.5).

In addition to Sobolev type inequalities, (1.2) can also be linked to some concentration inequalities. Letting for simplicity $\mu = \nu$, (1.2) is equivalent (see [BH]) to

$$\nu^n(A^h) \geq \nu\left(\left(-\infty, a + \frac{h}{4\sqrt{3}}\right]\right),\ h > 0, \tag{1.8}$$

where $a$ is chosen such that $\nu^n(A) = \nu((-\infty, a])$ and where $A \subset \mathbf{R}^n$ is an arbitrary Borel set. In this setting, Talagrand [T] (see also Maurey [M]) proved that

$$\nu^n(A + \sqrt{h}B_2 + hB_1) \geq \nu\left(\left(-\infty, a + \frac{h}{K}\right]\right),\ h > 0, \tag{1.9}$$

where $B_2$ and $B_1$ are respectively the $\ell^2$ and $\ell^1$ unit balls in $\mathbf{R}^n$ and where $K$ is a universal constant. Since $A^h = A + hB_2$, (1.9) is stronger than (1.8) for $h$ large. However, for $h$ small (which is important in obtaining sharp constant in Sobolev–type inequalities), (1.9) does not imply (1.8). It should nevertheless be noted here that (1.3) also involves a certain type of mixture of the $L^1$ and $L^2$ norms of the gradient.

A natural way to prove (1.2) is to establish its equivalent functional form (1.5) (with a dimension free constant $K(\mu)$). In turn, a natural way of proving (1.5) is to use an induction procedure on the dimension. However, the space $L^1$ does not seem adequate to perform this induction, and instead it is necessary to find an Orlicz space $L_N(X, \mu)$ for which this can be worked out. For this reason, the space $L_N$ generated by the function $N(x) = \sqrt{1 + x^2} - 1, x \in \mathbf{R}$, which behaves like $x^2$ for $|x|$ small and like $|x|$ for $|x|$ large will play an essential rôle. In particular, the inequality (1.3) corresponds to this choice of $N$. We are now ready to begin with some preliminaries.

## 2 A Generalization of Hölder's Inequality

Let $(\Omega, \mu)$ be a measure space and let $N : \mathbf{R} \to \mathbf{R}$ be a differentiable convex function.

**Lemma 2.1** *Let $f$ and $g$ be measurable functions on $\Omega$ such that*

$$\int N(g)d\mu \leq \int N(f)d\mu, \tag{2.1}$$

*then (provided all the written integrals exist)*

$$\int N'(f)gd\mu \leq \int N'(f)fd\mu. \tag{2.2}$$

**Proof.** It suffices to prove the result for $f$ and $g$ bounded and $\mu$ finite. First, by convexity,

$$\int N((1-t)f + tg)d\mu \leq (1-t)\int N(f)d\mu + t\int N(g)d\mu,\ 0 \leq t \leq 1. \tag{2.3}$$

Now, (2.3) becomes equality at $t = 0$ (and $t = 1$) and the left–hand side of (2.3) is a convex function of $t$ while the right–hand side is linear. Thus, at $t = 0$, the slope of the left–hand side of (2.3) is dominated by the slope of the right–hand side. Differentiating at $t = 0$ gives:

$$\int N'(f)(g-f)d\mu \leq \int N(g)d\mu - \int N(f)d\mu.$$

The lemma follows.

The proof above is due to A.V. Zhubr, and very elegantly replaces the original one. Let now $\|\cdot\|_p, p > 1$, denote the $L^p$–norm with respect to $\mu$, and let $q = p/(p-1)$. Applying Lemma 2.1, with $N(x) = |x|^p$, to $f = u^{1/(p-1)}, g = v$, where $u, v \geq 0$ are such that $\|u\|_q = 1, \|v\|_p = 1$, gives equality in (2.1); and (2.2) becomes:

$$\int uvd\mu \leq 1 = \|u\|_q\|v\|_p.$$

# 3 An Extension of Cheeger's Inequality

We return to the setting of the introductory section. Let also $N$ be a Young function, that is, $N : \mathbf{R} \to \mathbf{R}$ is even, non–negative such that $N(0) = 0$ and $N(x) > 0$ for all $x \neq 0$. Moreover, it is assumed that

$$C_N = \sup_{x>0} \frac{xN'(x)}{N(x)} < +\infty, \tag{3.1}$$

where $N'$ is a Radon–Nikodym derivative of $N$ (clearly, $C_N$ does not depend on the choice of $N'$).
We also denote by $L_N(X, \mu)$ the Orlicz space of functions $f$ such that

$$\|f\|_N = \inf\left\{\lambda > 0 : \mathbf{E}N\left(\frac{f}{\lambda}\right) \leq 1\right\} < +\infty.$$

Finally, and for simplicity, we write $\|\nabla f\|_N = \||\nabla f|\|_N$, while $m(f)$ denotes a median of $f$.

**Theorem 3.1** *Let $Is(\mu) > 0$. Then, for all functions $f$ which are Lipschitz on every ball in $X$ and such that $m(f) = 0$,*

$$\|f\|_N \leq \frac{C_N}{Is(\mu)}\|\nabla f\|_N, \tag{3.2}$$

$$\mathbf{E}N(f) \leq \mathbf{E}N\left(\frac{C_N}{Is(\mu)}|\nabla f|\right). \tag{3.3}$$

**Proof.** The isoperimetric constant $C = Is(\mu)$ is the optimal constant satisfying (3.2) when $N(x) = |x|$, i.e., such that

$$C\mathbf{E}|f| \leq \mathbf{E}|\nabla f|, \tag{3.4}$$

for all integrable, Lipschitz on every ball functions $f$ on $X$ with $m(f) = 0$. Indeed, following an argument of Ledoux [L], and via a co–area inequality in abstract spaces (see [BH])

$$\begin{aligned}
\mathbf{E}|\nabla f| &\geq \int_{-\infty}^{+\infty} \mu^+(f > t)d\mu \\
&\geq Is(\mu)\int_{-\infty}^{0}(1 - \mu(f > t))dt + Is(\mu)\int_0^{+\infty} \mu(f > t)dt \\
&= Is(\mu)\mathbf{E}|f|.
\end{aligned}$$

Therefore, $C \geq Is(\mu)$. On the other hand, applying (3.4) to a sequence of Lipschitz functions converging to $\mathbf{1}_A$ (again, for details see [BH]) gives $C \leq Is(\mu)$.

Now, let $f$ be a bounded, Lipschitz on every ball in $X$, function with $m(f) = 0$ and such that $\|f\|_N = 1$, that is such that $\mathbf{E}N(f) = 1$. Also, and without loss of generality, assume that $\mu(f = 0) = 0$ and that $N$ is differentiable (in case $\mu(f = 0) > 0$, one can just apply (3.2)–(3.3) to functions $f_k = f - \epsilon_k$, where $\epsilon_k \to 0$ as $k \to \infty$ are such that $\mu(f = \epsilon_k) = 0$). Let $f_1 = \max(f, 0)$ and $f_2 = \max(-f, 0)$. Then, $m(f_1) = m(f_2) = 0$, and thus $m(N(f_1)) = m(N(f_2)) = 0$. Applying (3.4) to $N(f_1)$ and $N(f_2)$ respectively gives

$$C\mathbf{E}N(f_1) \leq \mathbf{E}N'(f_1)|\nabla f_1| = \mathbf{E}N'(f)|\nabla f|\mathbf{1}_{(f>0)},$$

$$C\mathbf{E}N(f_2) \leq \mathbf{E}N'(f_2)|\nabla f_2| = -\mathbf{E}N'(f)|\nabla f|\mathbf{1}_{(f<0)}.$$

Therefore,

$$C\mathbf{E}N(f) = C\mathbf{E}N(f_1) - C\mathbf{E}N(f_2) \leq \mathbf{E}N'(f)|\nabla f|.$$

Next, applying Lemma 2.1 to $f$ and $g = |\nabla f|/\|\nabla f\|_N$ gives

$$\begin{aligned} C\mathbf{E}N(f) &\leq \|\nabla f\|_N \mathbf{E}N'(f)g \\ &\leq \|\nabla f\|_N \mathbf{E}N'(f)f \\ &\leq C_N\|\nabla f\|_N \mathbf{E}N(f). \end{aligned}$$

Hence, $C \leq C_N\|\nabla f\|_N$, and since $\|f\|_N = 1$, (3.2) follows. To get (3.3), it is enough to apply (3.2) to the functions $N_\alpha(x) = N(x)/\alpha, \alpha > 0$. Indeed, if $\|f\|_{N_\alpha} \geq 1$, then $\||\nabla f|/\lambda\|_{N_\alpha} \geq 1$, $\lambda = C_N/Is(\mu)$. Equivalently, if $\mathbf{E}N(f) \geq \alpha$, then $\mathbf{E}N\left(|\nabla f|/\lambda\right) \geq \alpha$. Theorem 3.1 follows.

**Remark 3.2** The inequalities (3.2)–(3.3) are Poincaré type inequalities. When, $N(x) = |x|^2$, and since $\|f - \mathbf{E}f\|_2 \leq \|f - m(f)\|_2$, (3.2) gives

$$C\|f - \mathbf{E}f\|_2 \leq \|\nabla f\|_2, \tag{3.5}$$

where $C \geq Is(\mu)/2$. Cheeger was the first to express the optimal constant $C$ in (3.5) in terms of the isoperimetric constant and so the inequality $C \geq$

$Is(\mu)/2$ bears his name. Cheeger's inequality has thus been extended in the following way: the optimal constant in (3.2)–(3.3) is such that

$$C \geq \frac{Is(\mu)}{C_N}. \tag{3.6}$$

For $N(x) = |x|^p$, the inequality (3.6) cannot be improved in terms of the isoperimetric constant. Indeed, taking $\mu = \nu$, (3.6) becomes equality as easily tested with the functions $\exp(\alpha x)$, $\alpha \to 1/p$.

**Remark 3.3** Note that (see Aida, Masuda and Shigekawa [AMS]) (3.5) implies that any Lipschitz function on $X$ has a finite exponential moment (of course, when $Is(\mu) > 0$).

# 4 Induction

**Lemma 4.1** *Let $C > 0$ be such that*

$$\int_X \sqrt{1+f^2}\,d\mu \leq \int_X \sqrt{1+C^2|\nabla f|^2}\,d\mu, \tag{4.1}$$

*for all Lipschitz functions $f$ on $X$ with $m(f) = 0$. Then,*

$$(\mu^n)^+(A) \geq \frac{1}{\sqrt{3}C}\mu^n(A)(1-\mu^n(A)), \tag{4.2}$$

*for all Borel sets $A \subset X^n$.*

**Proof.** If for all $x \in X$, $0 \leq f(x) \leq a$, then $|f(x) - m(f)| \leq a$ and

$$\sqrt{1+(f-m(f))^2} \geq 1 + K(a)(f-m(f))^2,$$

where $K(a) = (\sqrt{1+a^2}-1)/a^2$ is the optimal constant $K$ satisfying $\sqrt{1+t^2} \geq 1 + Kt^2$, for all $|t| \leq a$. Therefore,

$$\begin{aligned} \int_X \sqrt{1+(f-m(f))^2} &\geq 1 + K(a)\int_X (f-m(f))^2 d\mu \\ &\geq 1 + K(a)\mathbf{Var}(f). \end{aligned}$$

Thus, from (4.1),

$$1+K(a)\mathbf{Var}(f) \leq \int_X \sqrt{1+C^2|\nabla f|^2}d\mu, \tag{4.3}$$

for all Lipschitz functions $f$ on $X$ with $0 \leq f \leq a$. Now, by induction, (4.3) is extended to all Lipschitz functions $f : X^n \to [0,a]$ and it is proved that

$$1+L(a)\mathbf{Var}(f) \leq \int_X \sqrt{1+C^2|\nabla f|^2}d\mu, \tag{4.4}$$

where $L = L(a)$ is an arbitrary positive function such that

$$L(1+La^2) \leq K\left(\frac{a}{1+La^2}\right), L(a) \leq K(a). \tag{4.5}$$

To prove this induction step, take a Lipschitz function $f : X^{n+1} \to [0,a]$ and introduce the function

$$\alpha(y) = \int_{X^n} f(x,y)d\mu^n(x),\ y \in X.$$

Clearly, $\alpha : X \to [0,a]$ is Lipschitz and

$$|\nabla\alpha(y)| \leq \int_{X^n} |\nabla_y f(x,y)|d\mu^n(x),\ y \in X, \tag{4.6}$$

where $|\nabla_y f|$ is the modulus of gradient with respect to the coordinate $y$. Now, by our standing assumption, $|\nabla f|^2 = |\nabla_x f|^2 + |\nabla_y f|^2$, and thus

$$\int_{X^n} \sqrt{1+C^2|\nabla f|^2}d\mu^n(x) = \int_{X^n} \sqrt{(1+C^2|\nabla_x f|^2)+C^2|\nabla_y f|^2}d\mu^n(x) \tag{4.7}$$

The elementary inequality

$$\int \sqrt{u^2+v^2} \geq \sqrt{\left(\int u\right)^2 + \left(\int v\right)^2} \tag{4.8}$$

applied in (4.7) to $u = \sqrt{1+C^2|\nabla_x f|^2}$, $v = C|\nabla_y f|$ (keeping the coordinate $y$ fixed for a while) gives

$$\begin{aligned} &\int_{X^n} \sqrt{1+C^2|\nabla f|^2}d\mu^n(x) \\ \geq &\sqrt{\left(\int_{X^n} \sqrt{1+C^2|\nabla_x f|^2}d\mu^n(x)\right)^2 + C^2\left(\int_{X^n} |\nabla_y f|d\mu^n(x)\right)^2} \\ &\geq \sqrt{(1+L\mathbf{Var}_x(f))^2 + C^2|\nabla\alpha(y)|^2}, \end{aligned}$$

where the last inequality follows from the induction hypothesis as well as (4.6) and where $\mathbf{Var}_x(f)$ is the variance of $f$ with respect to $x \in X^n$. Next, since $\mathbf{Var}_x(f) \le a^2$, and since $\sqrt{1+t^2} - t, t > 0$, is decreasing in $t$ we get

$$\begin{aligned}
\int_{X^n} \sqrt{1 + C^2|\nabla f|^2} d\mu^n(x) \;&-\; (1 + L\mathbf{Var}_x f) \\
&\ge \sqrt{(1 + La^2)^2 + C^2|\nabla \alpha(y)|^2} - (1 + La^2) \\
&= (1 + La^2)\sqrt{1 + C^2|\nabla \alpha_1(y)|^2} - (1 + La^2), \quad (4.9)
\end{aligned}$$

where $\alpha_1 = \alpha/(1 + La^2)$. Integrating (4.9) over $y \in X$ and applying (4.3) to $\alpha_1$ and $a_1 = a/(1 + La^2)$ gives

$$\begin{aligned}
\int_{X^{n+1}} \sqrt{1 + C^2|\nabla f|^2} d\mu^{n+1}(x,y) \;&-\; \int_X (1 + L\mathbf{Var}_x(f)) d\mu(y) \\
&\ge (1 + La^2)(1 + K(a_1)\mathbf{Var}(\alpha_1)) - (1 + La^2) \\
&= (1 + La^2)K(a_1)\mathbf{Var}(\alpha_1) \\
&= \frac{1}{1 + La^2} K(a_1)\mathbf{Var}(\alpha).
\end{aligned}$$

In other words,

$$\int_{X^{n+1}} \sqrt{1 + C^2|\nabla f|^2} d\mu^{n+1} \ge 1 + L(a) \int_X \mathbf{Var}_x(f) d\mu(y) + \frac{K(a_1)\mathbf{Var}(\alpha)}{1 + L(a)a^2}. \quad (4.10)$$

Therefore, to finish the induction process via (4.10), it remains to show that

$$L(a) \int_X \mathbf{Var}_x(f) d\mu(y) + \frac{1}{1 + L(a)a^2} K(a_1)\mathbf{Var}(\alpha) \ge L(a)\mathbf{Var}(f). \quad (4.11)$$

Putting $\beta(y) = \int_{X^n} f^2(x,y) d\mu^n(x)$, we have

$$\begin{aligned}
\mathbf{Var}_x(f) &= \beta(y) - \alpha^2(y) \\
\mathbf{Var}(f) &= \int_X \beta(y) d\mu(y) - \left( \int_X \alpha(y) d\mu(y) \right)^2,
\end{aligned}$$

and (4.11) becomes

$$\begin{aligned} L(a)\left(\int\beta-\int\alpha^2\right) &+ \frac{1}{1+L(a)a^2}K(a_1)\mathbf{Var}(\alpha) \\ &\geq L(a)\left(\int\beta-\left(\int\alpha\right)^2\right). \end{aligned}$$

In turn, this is equivalent to

$$\frac{K(a_1)\mathbf{Var}(\alpha)}{(1+L(a)a^2)} \geq L(a)\mathbf{Var}(\alpha),$$

that is to

$$L(a)\left(1+L(a)a^2\right) \leq K\left(\frac{a}{1+L(a)a^2}\right).$$

But, by (4.5) this last inequality is true and under this condition, (4.4) is proved. Now, from (4.4) using the inequality $\sqrt{1+t^2} \leq 1+t$, we obtain

$$L(a)\mathbf{Var}(f) \leq C\int_{X^n}|\nabla f|d\mu^n,$$

for any Lipschitz function $f : X^n \to [0,a]$. That is, for every $f : X^n \to [0,1]$ and $a > 0$,

$$L(a)a\mathbf{Var}(f) \leq C\int_{X^n}|\nabla f|d\mu^n. \tag{4.12}$$

Applying (4.12) to a sequence of Lipschitz functions $f_k$ converging pointwise to the indicator function $\mathbf{1}_A$ so that to have $(\mu^n)^+(A) \geq \liminf\int_{X^n}|\nabla f_k|d\mu^n$, we get

$$(\mu^n)^+(A) \geq \frac{L(a)a}{C}\mu^n(A)(1-\mu^n(A)).$$

It just remains to show that

$$\sup_{L\ in\ (4.5)}\sup_{a>0} L(a)a \geq \frac{1}{\sqrt{3}}.$$

Let the function $L(a)$ behave like $w/a$ at infinity, so that $L(1+La^2) \to w^2$, $a/(1+La^2) \to 1/w$, as $a \to +\infty$. Therefore, (4.5) is fulfilled for all $a$ large enough if

$$w^2 < K\left(\frac{1}{w}\right), \quad w < 1, \tag{4.13}$$

since $K(a) \sim 1/a$ as $a \to +\infty$. But, the first inequality in (4.13) is equivalent to $\sqrt{1+t^2} - 1 > 1$ $(t = 1/w)$. In turn, this is equivalent to $t > \sqrt{3}$, i.e., $w < 1/\sqrt{3}$, and so the second inequality of (4.13) holds true.
Finally we get

$$\sup_{a>0} L(a)a \geq \sup_{w<1/\sqrt{3}} w = \frac{1}{\sqrt{3}}.$$

The lemma is proved.

# 5 Proof of Theorem 1.1

For the Young function $N(x) = \sqrt{1+x^2} - 1$, we have $C_N = 2$. Now, combine Theorem 3.1 and Lemma 4.1. By (3.3), the inequality (4.1) holds with $C = 2/Is(\mu)$, hence from (4.2),

$$(\mu^n)^+(A) \geq \frac{Is(\mu)}{2\sqrt{3}}\mu^n(A)(1-\mu^n(A)) \geq \frac{Is(\mu)}{4\sqrt{3}}\min(\mu^n(A), 1-\mu^n(A)).$$

Therefore, (1.2) follows. Next applying once more (3.3) to $(X^n, d_n, \mu^n)$ we have

$$\mathbf{E}N(f - m(f)) \leq \mathbf{E}N\left(\frac{8\sqrt{3}}{Is(\mu)}|\nabla f|\right), \tag{5.1}$$

for any Lipschitz on every ball function $f$ on $X^n$. If $0 \leq f \leq 1$, then $|f - m(f)| \leq 1$ and so $N(f - m(f)) \geq (f - m(f))^2/3$, therefore (5.1) gives

$$\frac{1}{3}\mathbf{Var}(f) \leq \mathbf{E}\sqrt{1 + \left(\frac{8\sqrt{3}}{Is(\mu)}\right)^2 |\nabla f|^2} - 1. \tag{5.2}$$

Now, note that for all $x \in \mathbf{R}$, $\sqrt{1+4x^2} - 1 \leq 2\min(|x|, x^2)$. Hence, the right–hand side of (5.2) is estimated by

$$2\mathbf{E}\min\left(\frac{4\sqrt{3}}{Is(\mu)}|\nabla f|, \left(\frac{4\sqrt{3}}{Is(\mu)}\right)^2 |\nabla f|^2\right)$$

$$\leq 96\mathbf{E}\min\left(\frac{1}{Is(\mu)}|\nabla f|, \frac{1}{Is^2(\mu)}|\nabla f|^2\right).$$

**Remark 5.1** Most likely the constant $K = 1/4\sqrt{3}$ in (1.2) is not optimal. In any case, in order to satisfy (1.2) for all measures $\mu$, it has to be less than 1. For individual measures, the optimal constant $K_\mu$ in (1.2) depends on $\mu$ and clearly satisfies $K_\mu \leq 1$. When $\mu$ is Gaussian, we have $K_\mu = 1$, as seen from the isoperimetric inequality in Gauss space. We do not know if there exist other probability distributions with this property.

**Remark 5.2** It is clear from the proof of Theorem 1.1, that for triples $(X_i, d_i, \mu_i)$, $i = 1, \cdots, m$, satisfying the hypotheses of Theorem 1.1, (1.2) takes the form

$$Is(\mu_1 \otimes \cdots \otimes \mu_m) \geq K \min_{1\leq i\leq m} Is(\mu_i),$$

where $K$ is an absolute constant. The inequality

$$Is(\mu_1 \otimes \cdots \otimes \mu_m) \leq \min_{1\leq i\leq m} Is(\mu_i),$$

is trivial. So, for product measures $\mu^n$, our results give Cheeger and Buser (see [L]) type inequalities independent of the dimension.

# 6 Proof of Theorem 1.2

When $n = 1$, this characterization is proved in [BH] and so the necessity part of the theorem does not require any proof. For the sufficiency, let $U : \mathbf{R} \to \mathbf{R}$ be the increasing map which transforms $\nu$ into $\mu$, and let $\|U\|_{Lip} < +\infty$. Let $\xi = (\xi_1, \cdots, \xi_n)$ be a random vector with distribution $\nu^n$, so that $\eta = (U(\xi_1), \cdots, U(\xi_n))$ has law $\mu^n$. Let $f : \mathbf{R}^n \to [0,1]$ be a Lipschitz function, then

$$|\nabla f o U| \leq \|U\|_{Lip}|(\nabla f)(U)|.$$

By the second part of Theorem 1.1, we then have (remembering that $Is(\nu) = 1$) that

$$\begin{aligned} K\mathbf{Var} f(\eta) &\leq \mathbf{E}|\nabla f o U(\xi)| \\ &\leq \|U\|_{Lip}\mathbf{E}|\nabla f(U(\xi))| = \|U\|_{Lip}\mathbf{E}|\nabla f(\eta)|. \end{aligned} \tag{6.1}$$

Approximating the indicator function $\mathbf{1}_A$ of a Borel set $A \subset \mathbf{R}^n$ by a sequence of Lipschitz functions, it follows from (6.1) that

$$(\mu^n)^+(A) \geq \frac{K}{\|U\|_{Lip}} \mu^n(A)(1 - \mu^n(A)).$$

This gives (1.5) for any Lipschitz function $f : \mathbf{R}^n \to \mathbf{R}$ for which $\mathbf{E}f(\eta) = 0$ (see [BH]).

# 7 Poincaré Type Inequalities in Product Spaces

Here, we use Theorem 1.1, to obtain the statements of Theorem 3.1 in the $n$–dimensional space $(X^n, d_n, \mu^n)$ under the "more natural" assumption $\mathbf{E}f = 0$. Again, let $N$ satisfy the same hypothesis as before, let $\|\cdot\|_N$ denote the norm in the Orlicz space $L_N(X^n, \mu^n)$, and let $\mathbf{E}$ be the expectation with respect to $\mu^n$.

**Theorem 7.1** *For any Lipschitz on every ball function $f$ on $X^n$ with $\mathbf{E}(f) = 0$,*

$$\|f\|_N \leq \frac{8\sqrt{3}C_N}{Is(\mu)} \|\nabla f\|_N. \tag{7.1}$$

*In particular,*

$$\mathbf{E}N(f) \leq \mathbf{E}N\left(\frac{8\sqrt{3}C_N}{Is(\mu)} |\nabla f|\right). \tag{7.2}$$

**Proof.** On $X^n \times X^n$, let $g(x, y) = f(x) - f(y), x, y \in X$. Since, $m(g) = 0$ with respect to $\mu^{2n}$, applying Theorem 3.1 and (1.2) gives:

$$\|g\|_N \leq \frac{4\sqrt{3}C_N}{Is(\mu)} \|\nabla g\|_N,$$

where now, $\|\cdot\|_N$ denotes the norm in $L_N(X^{2n}, d_{2n}, \mu^{2n})$. Since, $|\nabla g(x, y)| = \sqrt{|\nabla f(x)|^2 + |\nabla f(y)|^2} \leq |\nabla f(x)| + |\nabla f(y)|$, we get $\|g\|_N \leq 2\|f\|_N$. Thus,

$$\|g\|_N \leq \left\| \frac{8\sqrt{3}C_N}{Is(\mu)} |\nabla f| \right\|_N. \tag{7.3}$$

Applying (7.3) to the functions $N_\alpha(t) = N(t)/\alpha, t \in \mathbf{R}, \alpha > 0$, one easily obtains

$$\int_{X^n}\int_{X^n} N(g(x,y))d\mu^n(x)d\mu^n(y) \le \int_{X^n} N\left(\frac{8\sqrt{3}C_N}{Is(\mu)}|\nabla f|\right) d\mu^n. \qquad (7.4)$$

But, by the convexity of $N$, $\int\int N(f(x)-f(y)) \ge \int N(f(x) - \int f)$. This gives (7.1). In turn, applying (7.1) to the functions $N_\alpha$ gives (7.2) and the theorem is proved.

**Remark 7.2** When $N(x) = |x|^p, p \ge 1$, then $C_N = p$, and (7.1) becomes

$$\|f - \mathbf{E}f\|_p \le \frac{8\sqrt{3}}{Is(\mu)} p\|\nabla f\|_p, \qquad (7.5)$$

where the constant is of sharp order in $p$. This can be tested for the measure $\nu$ on the function $f(x) = \exp \alpha x, x \in \mathbf{R}$, letting $\alpha \to 1/p$. When $\mu$ is Gaussian on $X = \mathbf{R}$, (7.3) with a better constant (of order $\sqrt{p}$) can be found in Pisier [P].

**Remark 7.3** In (7.1)–(7.2), we have not been very careful about the constants and just tried for example in (7.6), to find the right order in $p$. For inequalities such as (7.1)–(7.2), where the mean replaces the median, the isoperimetric constant $\inf(\mu^n)^+(A)/2\mu^n(A)(1-\mu^n(A))$ (equivalent to (1.1)) is a little more precise.

# 8 Khinchine–Kahane Type Inequalities

Let $\xi_1, \cdots, \xi_n$, be i.i.d. random variables on the real line $\mathbf{R}$, with law $\mu$ and such that $Is(\mu) > 0$. As noted in Remark 3.3, this last condition implies that $\mu$ has finite first moment. Let us also assume that $\mathbf{E}\xi_1 = 0$, and that $\xi_1 \ne 0$, a.s.

Let $N$ be a Young function such that

$$K_N = \|\xi_0\|_N < +\infty,$$

where $\xi_0$ is a random variable which has a double exponential distribution.

**Theorem 8.1** *There exists a finite positive constant $C = C(N, \mu)$ such that for any Banach space $(B, \|\cdot\|_B)$ and vectors $v_1, \cdots, v_n \in B$,*

$$\|\xi\|_N \leq C\|\xi\|_1, \tag{8.1}$$

*where $\xi = \|\xi_1 v_1 + \cdots + \xi_n v_n\|_B$.*

In (8.1), one can take $C = 2 + \sqrt{2} Is(\mu) K_N / (4\sqrt{3}\mathbf{E}|\xi_1 - m(\xi_1)|)$. When $N(x) = |x|^p$ and $\mu$ is Gaussian, (8.1) is well known (see e.g., [P, p.179]).

**Proof.** The inequality (1.2)

$$(\mu^n)^+(A) \geq \frac{Is(\mu)}{4\sqrt{3}} \min(\mu^n(A), 1 - \mu^n(A)),$$

$A \subset \mathbf{R}^n$, can easily be integrated ([BH]) to give

$$\mu^n(A^h) \geq R_{Kh}(\mu^n(A)),\ h > 0,\ K = \frac{Is(\mu)}{4\sqrt{3}}, \tag{8.2}$$

where the function $R_h$ is defined by $R_h(p) = \nu((-\infty, a+h]), p = \nu(-\infty, a]$, $a \in \mathbf{R}$. In particular, when $\mu^n(A) \geq 1/2$, (8.2) gives

$$1 - \mu^n(A^{Kh}) \leq \frac{1}{2}e^{-h} = \mathbf{P}(\xi_0 > h). \tag{8.3}$$

Therefore, for functions $f : \mathbf{R}^n \to \mathbf{R}$ with $\|f\|_{Lip} < +\infty$, applying (8.3) to the sets $A = \{f < t\}$, we have

$$\mu^n(|f - m(f)| > K\|f\|_{Lip} h) \leq \mathbf{P}(|\xi_0| > h).$$

By the very definition of the Orlicz norm, this gives

$$\|f - m(f)\|_N \leq K\|f\|_{Lip}\|\xi_0\|_N = KK_N\|f\|_{Lip}. \tag{8.4}$$

Now, let $f(x) = \sup \sum_{k=1}^n < w, v_k > x_k$, where the sup is taken over the unit ball of the dual $B^*$. The function $f$ is such that

$$\|f\|_{Lip}^2 \leq \sigma^2 = \sup_{\|w\|_{B^*}} \sum_{k=1}^n (< w, v_k >)^2,$$

and moreover it has (with respect to $\mu^n$) the same distribution as $\xi$. Hence, (8.4) can be rewritten as

$$\|\xi - m(\xi)\|_N \leq KK_N\sigma. \tag{8.5}$$

To estimate $\sigma$ via $\mathbf{E}\xi$, we make use of a recent result in [B]: if $\xi_1, \cdots, \xi_n$ are zero mean, independent random variables, with $\mathbf{E}|\xi_k - m(\xi_k)| \geq 1$, then

$$\mathbf{E}|a_1\xi_1 + \cdots + a_n\xi_n| \geq \mathbf{E}|a_1\eta_1 + \cdots + a_n\eta_n|,$$

for all $a_1, \cdots, a_n \in \mathbf{R}$, where $\eta_1, \cdots, \eta_n$ are i.i.d. Bernoulli random variables. Thus, using Khinchine inequality with the optimal constant (see Szarek [S]), it follows that

$$\frac{\sqrt{2}}{\mathbf{E}|\xi_1 - m(\xi_1)|}\mathbf{E}|a_1\xi_1 + \cdots + a_n\xi_n| \geq \sqrt{a_1^2 + \cdots + a_n^2}. \tag{8.6}$$

Applying (8.6) to $a_k =< w, v_k >$ and using (8.5), we get

$$\begin{aligned}\left(\sum_{k=1}^{n}(< w, v_k >)^2\right)^{1/2} &\leq \frac{\sqrt{2}}{\mathbf{E}|\xi_1 - m(\xi_1)|}\mathbf{E}\|v_1\xi_1 + \cdots + v_n\xi_n\|_B \\ &= \frac{\sqrt{2}}{\mathbf{E}|\xi_1 - m(\xi_1)|}\|\xi\|_1. \end{aligned} \tag{8.7}$$

Thus, $\sigma \leq \sqrt{2}\|\xi\|_1/\mathbf{E}|\xi_1 - m(\xi_1)|$. Finally, using the elementary inequality $m(\xi) \leq 2\mathbf{E}\xi$, it follows from (8.6) and (8.7) that

$$\|\xi\|_N \leq m(\xi) + KK_N\sigma \leq \left(2 + \frac{\sqrt{2}KK_N}{\mathbf{E}|\xi_1 - m(\xi_1)|}\right)\|\xi\|_1.$$

# References

[AMS] Aida, S., Masuda, T., Shigekawa, I. Logarithmic Sobolev inequalities and exponential integrability. To appear in J. Func. Analysis.

[B] Bobkov, S.G. Some extremal properties of Bernoulli distribution. To appear in: Probab. Theory Appl. (In Russian).

[BH] Bobkov, S.G., Houdré, C. Some connections between Sobolev-type inequalities and isoperimetry. Preprint (1995).

[BU] Borovkov, A.A., Utev, S.A. On an inequality and a related characterization of the normal distribution. Probab. Theory Appl. **28** (1983), 209–218 (In Russian).

[C] Cheeger, J. A lower bound for the smallest eigenvalue of the Laplacian. Problems in Analysis, Symposium in honor of S.Bochner. 195–199, Princeton Univ. Press, Princeton (1970).

[CL] Chen, L.H.Y., Lou, J.H. A characterization of probability measures which admit Poincaré inequalities. Preprint.

[G] Gross, L. Logarithmic Sobolev inequalities and contractivity properties of semigroups. Varenna, (1992). Lect. Notes Math. **1563** (1993), 54–88.

[L] Ledoux, M. A simple analytic proof of an inequality by P.Buser. Proc. Amer. Math. Soc. **121** (1994), 951–959.

[M] Maurey, B. Some deviation inequalities. Geom. and Funct. Anal. **1** (1991), 188–197.

[P] Pisier, G. Probabilistic methods in the geometry of Banach spaces. Probability and Analysis, Varenna (Italy) 1985. Lect. Notes Math. **1206** (1986), 167–241.

[S] Szarek, S.J. On the best constant in Khinchine inequality. Studia Math. **58** (1976), 197–208.

[T] Talagrand, M. A new isoperimetric inequality and the concentration of measure phenomenon. Israel Seminar (GAFA), Springer Verlag, Lect. Notes in Math. **1469** (1991), 94–124.

Sergei. G. Bobkov
Department of Mathematics
Syktyvkar University
167001 Syktyvkar
Russia

Christian Houdré
Center for Applied Probability
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30322, USA